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1987 J. Phys. A: Math. Gen. 20 L143

(http://iopscience.iop.org/0305-4470/20/3/005)

J. Phys. A: Math. Gen. 20 (1987) L143-L146. Printed in the UK

LETTER TO THE EDITOR

A specific model which realises the stochastic quantisation prescriptions

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Received 27 October 1986

Abstract. In the present letter we show how the phase space quantisation approach can be fulfilled via the collective behaviour of quantum mechanical particles on a cubic lattice.

One of the most interesting approaches to the quantisation problem is that which formulates itself on the phase space of the physical system under consideration. Such a formulation is based on the observation made by several authors [1-5] who have suggested a quantisation mapping adhering to the form

$$\hat{f} = \mathrm{i}X_f + f - p\,\partial f/\partial p - q\,\partial f/\partial q \qquad (1)$$

where X_f is a tangent vector on phase space associated with the function f for some vector field X [6, 7].

Specific realisations of quantisation mappings resulting from the general prescription given by (1) are the following.

(i) Van Hove's [1, 5] mapping

$$\hat{Q} = q + \mathrm{i}\hbar\,\partial/\partial p \qquad (2a)$$

$$\hat{P} = -\mathrm{i}\hbar\,\partial/\partial q \qquad (2b)$$

or its improved form [8]

$$\hat{Q} = \tfrac{1}{2}q + \mathrm{i}\hbar\,\partial/\partial p \qquad (3a)$$

$$\hat{P} = -2\mathrm{i}\hbar\,\partial/\partial q. \qquad (3b)$$

(ii) The symmetric quantisation mapping [2, 3, 4, 8]

$$\hat{Q} = \mathrm{i}\hbar\,\partial/\partial p + \tfrac{1}{2}q \qquad (4a)$$

$$\hat{P} = -\mathrm{i}\hbar\,\partial/\partial q + \tfrac{1}{2}p. \qquad (4b)$$

There are various reasons why the above prescriptions might be advantageous with respect to the conventional quantisation mapping (see references [1-8] and also Chernoff [9]).

In recent papers [8, 10, 11] we have arrived at several new realisations regarding phase space quantisation (for non-relativistic systems), on the basis of certain connections we were able to make among the various approaches contained in [1-7]. The basic picture emerging from our work so far is consistent with the view that phase space quantisation amounts to a coherent superposition of conventional quantum states. This is implicit in the approach to quantisation advocated by Prugovečki and collaborators [5, 12]. Their basic idea is to exploit the Van Hove quantisation rule (2*a*), (2*b*) in order to attain optimum localisability of a quantum mechanical particle. This, in turn, leads to a stochastic formulation of quantum mechanics.

From our viewpoint a scheme which employs states of minimum uncertainty, such as the one of Prugovečki, should exhibit common characteristics with schemes employing coherent states. In fact, we have been able to show [8] that the stochastic approach can be reformulated in terms of a space of states on phase space which employs for its inner product a density function $\rho(q, p)$ of the form

$$\rho(q, p) = \exp[-a(q^2 + p^2)] \tag{5}$$

where a is a constant to be fixed by appropriate normalisation. Such a density function has been introduced by Bargmann [13] in his work on coherent states.

In the present letter we shall explicitly display the coherence content of the stochastic quantisation scheme through a specific model which is widely employed in solid state physics. In particular, we shall consider a cubic crystal lattice on each site of which resides an electron. Our aim is to show that the description of the system as a whole is attained within the context of the stochastic approach to quantisation. Specifically, we shall establish a connection between the Bloch functions of solid state physics, employed to describe the electronic lattice, and the phase space functions of stochastic quantisation. Prugovečki's stochastic quantisation scheme begins with the observation that any quantisation prescription associated with the mapping (2*a*), (2*b*) is highly reducible on the space of square integrable functions on phase space Γ. To this end he employs a function $\xi(\boldsymbol{x})$ which generates a family $\{\xi_{p,q}\}$ through its Galilean phase space translations, such that $\{\xi_{p,q}(\boldsymbol{x})\}$ constitutes a continuous resolution of the identity in $L^2(R^3)$. Formally we write

$$\int_\Gamma |\xi_{p,q}\rangle \, \mathrm{d}q \, \mathrm{d}p \langle \xi_{p,q}| = \mathbb{1}. \tag{6}$$

In this way one arrives at a subspace Γ_ξ of phase space [12], the square integrable functions which provide a space of states $L^2(\Gamma_\xi)$ carrying an irreducible representation of Van Hove's mapping (2*a*), (2*b*). Presumably, other mappings resulting from (1) can be similarly accommodated by appropriate phase space constructions.

The fact that the family $\{\xi_{p,q}\}$ ı solves the identity enables us to expand the phase space state function $\Psi(p, q) \in L^2(\text{ı}_\xi)$ as follows:

$$\Psi(p, q) = \int \Psi(\boldsymbol{x}) \xi_{q,p}(\boldsymbol{x}) \, \mathrm{d}^3 \boldsymbol{x}. \tag{7}$$

Now, according to what we have said before, $\xi_{q,p}(\boldsymbol{x})$ results from Galilean translations of a generating function $\xi(\boldsymbol{x})$. Thus we have

$$\xi_{p,q}(\boldsymbol{x}) = \exp(\mathrm{i}\boldsymbol{p} \cdot \boldsymbol{x}) \xi(\boldsymbol{x} - \boldsymbol{q}). \tag{8}$$

We note, in passing, that similarly constructed functions have also been employed by Davies [14] in order to discuss observables on phase space.

Clearly, there is a multitude of choices for the generating functions $\xi(\boldsymbol{x})$. In this letter we shall arrive at particular choices for $\xi(\boldsymbol{x})$ dictated from a solid state theoretical model. Specifically, we shall consider an electronic cubic lattice which is conventionally described by the so-called Bloch functions.

In order to introduce the lattice structure into our considerations we give the following structure to $\Psi(\boldsymbol{x})$, entering (7),

$$\Psi(\boldsymbol{x})=\sum_{\boldsymbol{l}}\delta(\boldsymbol{x}-\boldsymbol{l}) \tag{9}$$

where $\Sigma_{\boldsymbol{l}}$ extends over the points of the cubic lattice in space. Substituting in (7) we obtain

$$\Psi(p,q)=\sum_{\boldsymbol{l}}\xi_{p,q}(\boldsymbol{l}) \tag{10}$$

and by appealing to (8) we finally obtain

$$\Psi(p,q)=\sum_{\boldsymbol{l}}\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{l}}\xi(\boldsymbol{l}-\boldsymbol{q}). \tag{11}$$

Let us now suppose that $\xi(\boldsymbol{x})$ corresponds to a wavefunction for a given particle on the lattice. It will have, in general, an expansion in terms of an orthonormal set $\{\phi_n(\boldsymbol{x})\}$ of eigenfunctions of a certain Hermitian operator, say the Hamiltonian. In other words we can write

$$\xi(\boldsymbol{x})=\sum_n c_n\phi_n(\boldsymbol{x}) \tag{12}$$

whereupon (11) becomes

$$\Psi(p,q)=\sum_n\sum_{\boldsymbol{l}}c_n\,\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{l}}\phi_n(\boldsymbol{l}-\boldsymbol{q}). \tag{13}$$

For each given n we may write

$$\Psi_n(p,q)=c_n\sum_{\boldsymbol{l}}\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{l}}\phi_n(\boldsymbol{l}-\boldsymbol{q}). \tag{14}$$

In the above expression we recognise the Bloch functions which describe the electronic lattice. Conventionally these functions are given as follows:

$$\Psi_{p,n}(q)=\frac{1}{\sqrt{N}}\sum_{\boldsymbol{l}}\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{l}}\phi_n(\boldsymbol{l}-\boldsymbol{q}) \tag{15}$$

where N is the number of points in the lattice and the set $\phi_n(\boldsymbol{x})$ is complete and normalised, but not necessarily orthogonal.

It is certainly interesting that a highly general consideration pertaining to phase space quantisation has resulted in a formulation which is widely employed in an applied branch of modern physics. Clearly, the model input to our considerations comes through relation (9), which introduces a particle at each site of the lattice. From our viewpoint the above result is consistent with the coherence interpretation of stochastic quantisation which, for the particular model examined, enters via the collective behaviour of the particles on the lattice.

We are now in a position to comment on the nature of the functions $\xi(\boldsymbol{x})$ which generate the space carrying an irreducible representation of Van Hove's commutation relations. It becomes obvious from (12) that $\xi(\boldsymbol{l}-\boldsymbol{q})$ is a square integrable function which coincides with a wavefunction for the particle situated at the lattice point $\boldsymbol{l}$. We thereby realise that the mathematically meaningful space $L^2(\Gamma_\xi)$ which carries an irreducible representation of Van Hove's commutation relations is generated from the physically meaningful function $\xi(\boldsymbol{x})$ describing the state of a given particle on the lattice.

Actually, the description of $\xi(x)$ in terms of a single particle on the lattice is best attained through the so-called Wannier functions. The latter are truly localised about each electron and at the same time they constitute a complete orthonormal basis. This property of the Wannier functions is connected to the fact that they are associated with an effective Hamiltonian which localises the effects of the whole lattice on each particle. Thus our interpretation of the $\xi(x)$ as a function referring to a single particle on the lattice is fully realisable.

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